Complex contact bundles and their applications to theory of the space-time

Thoughts on advanced geometry of the bundle of skies

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Abstract

The paper defines concept of a complex contact bundle $\Omega \to Q$ where Ω is endowed with a holomorphic contact structure but Q is smooth only. Some cases produce an "anisotropic" complex structure on Q or some its points – where multiplication by i in the tangent space is non-linear, albeit linear within 1-subspaces. There is an example where Q is a Grassmanian in symplectic geometry. A generalization of such bundles to Ω endowed with certain complex tangent subbundle is applicable to Lorentzian geometry; namely, as a space where the bundle of skies can be embedded.

Note. This text is not about to become a peer-reviewed publication and will be superseded by newer papers in 2020. It is retained mainly to document development of the theory.

1 Complex bundles

1.1 Opening definitions and problems

Definition 1. A *complex bundle* $\Omega \to Q$ is a complex manifold Ω endowed with such smooth mapping $\mathrm{q} : \Omega \to Q$ onto certain real manifold[1] Q that for any $z \in \Omega$:

[1] Not necessarily C^∞. Specific smoothness conditions will be formulated on applications.

- $\mathrm{im}(d\mathrm{q}|_z) = T_{q(z)}Q$,
- $\ker(d\mathrm{q}|_z)$ is a complex subspace in $T_{q(z)}\Omega$.

The latter is, in other words, a requirement for all preimages $\Omega_q := \mathrm{q}^{-1}(q)$, where $q \in Q$, to be complex submanifolds.

Remark. Note that Q needs not to have any (almost) complex structure at all, because no requirement exists on dependence of the bundle on *the base* (beside smoothness). Besides such obvious covering examples as $\mathbb{C}\mathbf{P}^1 \simeq S^2 \xrightarrow{2:1} \mathbb{R}\mathbf{P}^2$, the example from the subsection 1.2 will demonstrate that a projection onto Q having *connected* fibers may result in no complex structure.

Definition 2. A *complex contact bundle* $\Omega \to Q$ where $\dim_{\mathbb{C}} \Omega = 2m+1$ for a natural m, is a complex bundle where Ω is endowed with a holomorphic contact structure.

Remark. The most obvious example is the projective contangent bundle $\mathbf{P}T'Q$, where Q is a complex manifold, with the standard contact form $\Theta = p_k\, dq^k$. For it, any complex hypersurface in Q lifts to a Legendrian submanifold[2] in $\mathbf{P}T'Q$.

Is it possible to have all Ω_q Legendrian but without a complex structure on Q? Are there interesting contact examples where Ω_q isn't Legendrian for some $q \in Q$?

We shall demonstrate that for $m = 1$ and $\dim Q = 4$ the answer to the latter question is positive, and geometry induced by the complex contact bundle in some cases "looks like" an (almost) complex structure on Q.

Definition 3. A *Legendrian point* $q \in Q$ of a complex contact bundle $\Omega \to Q$ is a point where Ω_q is a Legendrian submanifold in Ω. A *complex Legendrian bundle* is a complex contact bundle with $\dim Q = 2m+2$ where all points in the receiving manifold Q are Legendrian.

1.2 The oriented 2d Grassmanian

Definition 4. The oriented Grassmanian $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^n)$ is the manifold of oriented homogeneous planes in $\mathbb{R}^n$.

[2] Remind that a *Legendrian submanifold* Σ in a $(2m+1)$-dimensional manifold $\mathcal{C}$ having a contact structure—either real or complex—is an m-dimensional submanifold tangent to the contact structure, that is, $T\Sigma \subset T\mathrm{c}\,\mathcal{C}|_\Sigma$, where $T\mathrm{c}$ denotes the contact structure – a $2m$-dimensional subbundle in $T\mathcal{C}$.

Example. A complex bundle, the *complex-forgetful plane map* $\mathrm{r} : \mathbb{C}\mathbf{P}^{n-1} \setminus \mathbb{R}\mathbf{P}^{n-1} \to \widetilde{\mathrm{Gr}}_2(\mathbb{R}^n)$ is $\mathrm{r}(\mathrm{P}\mathbf{z}) := \langle \operatorname{Re}\mathbf{z}, \operatorname{Im}\mathbf{z}\rangle_{\mathbb{R}}$ where the vector $\mathbf{z}$ belongs to $\mathbb{C}^n$ but isn't a complex multiple of any element of $\mathbb{R}^n$, $\mathrm{P}\mathbf{z}$ denotes the projective image of $\mathbf{z}$, and $\langle\cdots\rangle_{\mathbb{R}}$ is the real linear span. The orientation for the spanned two-dimensional subspace follows the order of two vectors under $\langle\cdots\rangle_{\mathbb{R}}$.

Obviously, this element of the Grassmanian doesn't change under multiplication of $\mathbf{z}$ by non-zero complex numbers.

Remark. Fibers of the bundle are complex discs. In terms of $\mathbb{C}\mathbf{P}^{n-1}$ these are halves of projective lines and their boundaries lie in the real space $\mathbb{R}\mathbf{P}^{n-1}$.

Digression. For $n = 4$ it's an exercise to show that the complex structure on $\mathbb{C}\mathbf{P}^{n-1} \setminus \mathbb{R}\mathbf{P}^{n-1}$ is compatible with the pseudo-Riemannian (conformal) structure on the two-dimensional Grassmanian[3] in the sense of

$$\Psi(\mathrm{r}_*(i\,v)) = \Psi(\mathrm{r}_*(v)) \text{ where } v \in T(\mathbb{C}\mathbf{P}^3 \setminus \mathbb{R}\mathbf{P}^3).$$

Remind that "$\cdots_*$" denotes the pushforward by a differentiable map.

1.3 A contact bundle over the Grassmanian

Consider a skew bilinear (symplectic) form ω on $\mathbb{R}^4$, such as this defined by the matrix

$$\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

where I_2 is the identity 2×2 matrix. It induces a contact structure on both respective real and complex projective spaces, namely the real structure $\omega(\mathbf{x}, d\mathbf{x}) = 0$ where $\mathbf{x} \in \mathbb{R}^4$ on $\mathbb{R}\mathbf{P}^3$ and the complex holomorphic structure $\omega(\mathbf{z}, d\mathbf{z}) = 0$ where $\mathbf{z} \in \mathbb{C}^4$ on $\mathbb{C}\mathbf{P}^3$.

Remark. This contact structure is preserved under multiplication of ω by non-zero numbers. It will have implications for the subsection 2.2.

Example. The complex-forgetful plane map $\mathrm{r} : \mathbb{C}\mathbf{P}^3 \setminus \mathbb{R}\mathbf{P}^3 \to \widetilde{\mathrm{Gr}}_2(\mathbb{R}^4)$ defines a complex contact bundle. Its Legendrian points are exactly isotropic (totally null) planes of ω which form a hypersurface in $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^4)$.

[3] Assuming orientation on $\mathbb{R}^4$. This structure results from intersections of planes.

1.4 Legendrian points

Theorem 1. *For a complex contact bundle* $\Omega \to Q$, $\dim Q = 2m + 2$ *and any Legendrian point* $q \in Q$, *the map* $\mathrm{P}\tau|_{\Omega_q} : \Omega_q \to \mathbf{P}(\mathbb{C} \otimes T'_q Q)$ *is a holomorphic immersion, where a complex-valued 1-form* τ*—representing the contact structure on* Ω *as* $T\mathrm{c}\,\Omega = \ker \tau$*—is a holomorphic section of* $(\ker d\mathrm{q})^{\perp} \subset T'\Omega$ *over some neighborhood in* Ω *and "*$(\cdots)^{\perp}$*" means "complex-linear functionals nullifying all the subspace". Moreover, if* $m = 1$*, then* $\mathrm{P}(\mathrm{q}_* \,(T\mathrm{c}^{1,0}\,\Omega|_{\Omega_q})) : \Omega_q \to \mathbf{P}(\mathbb{C} \otimes T_q Q)$ *is also a holomorphic immersion.*

Strictly speaking, a local form representing the contact structure should be a local section of $T'\Omega$, but—because the fiber Ω_q is Legendrian—we restrict its values to the dual horizontal bundle $(\ker d\mathrm{q})^{\perp} \simeq \mathbb{C} \otimes \mathrm{q}^*(T'Q)$. Here "$\mathrm{q}^*$" denotes the pullback by a differentiable map.

Proof. Both mappings are holomorphic by construction. The "$\mathrm{P}\tau$ immersion" part follows from maximal non-integrability of τ. The "$\mathrm{P}(\mathrm{q}_*\ T\mathrm{c})$ immersion" part follows from the $\mathrm{P}\tau$ immersion and the fact that $\mathrm{Re}\langle\tau\rangle_{\mathbb{C}}$ is exactly the two-dimensional space that nullifies all real contact vectors in Ω. $\square$

Remark. Note that $\dim_{\mathbb{C}} \Omega_q = m$, the same dimension the projectivisation of the space of $(1,0)$-forms could have were Q an (almost) complex manifold. The same for $(1,0)$-(vector fields) for $m = 1$. Hence, a complex Legendrian bundle can serve as an "anisotropic complex structure" for even-dimensional real manifolds. If the real part of the Q-projection of $\lambda\tau$, $\lambda \in \mathbb{C}^{\times}$ in some point of Ω is a "real covector multipliable by i", then a neighborhood in $T'Q$ will possess the same operation as well. The same for vectors from TQ ($\dim Q = 4$) resulting from projection of contact vectors in Ω. In the subsection 1.6 we shall see an "anisotropic CR structure" suitable also for odd-dimensional real manifolds.

1.5 Complex contact structure and vector fields

For any complex manifold M, its complex structure can be expressed in terms of the $T^{1,0}M$ subbundle in the complexified real tangent bundle $\mathbb{C} \otimes T_{\mathbb{R}}M$. For a $(2m+1)$-dimensional complex contact manifold $\mathcal{C}$, $m \geq 1$, it's possible to define both complex and contact structures at once with the subbundle $T\mathrm{c}^{1,0}\,\mathcal{C}$ of the rank $2m$. Although such definition can be understood as a special case of *almost* CR structure, it is manifestly not an [integrable] CR

(Cauchy–Riemann) structure because $[Tc^{1,0}\,\mathcal{C},\, Tc^{1,0}\,\mathcal{C}] = T^{1,0}\mathcal{C}$ where $[\cdot,\cdot]$ is the Lie bracket of vector fields a.k.a. commutator – that is, $Tc^{1,0}\,\mathcal{C}$ is not involutive.

Remark. If ϑ is the tautological contact 1-form (not important whether complex or real), acting from contact vector fields to the bundle "normal" to the structure, then its external derivative $d\vartheta$ can be defined as an antisymmetric bilinear form of the same domain and range, and corresponds to the Lie bracket via the identity

$$\vartheta[U,V] = (d\vartheta)(U,V).$$

For the complex case contact vector fields are sections of $Tc^{1,0}\,\mathcal{C}$ and the "normal bundle" is $T^{1,0}\mathcal{C}/\,Tc^{1,0}\,\mathcal{C}$.

For $m = 1$, reduction of complex contact structures to a special kind of vector subbundles permits for a generalization to be introduced by the following

Definition 5. A smooth manifold $\mathfrak{X}$ of unspecified dimension endowed with a smooth subbundle $T^{1,0}_{\mathrm{II}}\mathfrak{X} \subset \mathbb{C}\otimes T\mathfrak{X}$ of the rank 2 is a *quasi complex* ("Q$\mathbb{C}$" for short) *contact* manifold if $T^{1,0}_{\mathrm{III}}\mathfrak{X} := [T^{1,0}_{\mathrm{II}}\mathfrak{X}, T^{1,0}_{\mathrm{II}}\mathfrak{X}]$ defines a subbundle in $\mathbb{C}\otimes T\mathfrak{X}$ of the rank 3.

By the Lie bracket of tangent subbundles we understand the union of all graphs of Lie brackets of smooth sections of the specified subbundles.

Remark. This definition covers diverse objects, from the true three-dimensional contact structure to some type of non-integrable two-complex-dimensional almost complex manifolds.

Remark. For $\mathfrak{X}$ endowed with an arbitrary smooth two-dimensional $T^{1,0}_{\mathrm{II}}\mathfrak{X} \subset \mathbb{C}\otimes T\mathfrak{X}$ computing $[T^{1,0}_{\mathrm{II}}\mathfrak{X}, T^{1,0}_{\mathrm{II}}\mathfrak{X}]$ may add no more than one complex dimension to the subbundle due to the multiplication by a scalar function property of the Lie bracket and its skew symmetry. Hence, it's reasonable to consider the "almost contact" case for $\mathfrak{X}$, where in general points there is a Q$\mathbb{C}$ structure, but there are some degeneracies and, possibly, singularities around which $T^{1,0}_{\mathrm{III}}\mathfrak{X}$ cannot be defined continuously.

Definition 6. A *Legendrian complex curve* Σ in a Q$\mathbb{C}$-contact manifold $\mathfrak{X}$ is such smooth two-dimensional submanifold that $T^{1,0}_{\mathrm{II}}\mathfrak{X} \cap \mathbb{C}\otimes T\Sigma$ is a complex line bundle (that is, a vector bundle of the rank 1) over Σ.

Remark. For the $m = 1$ holomorphic contact case these submanifolds are namely Legendrian complex curves in the usual sense.

Remark. The definition can be generalized to the "almost contact" case, of a manifold $\mathfrak{X}$ endowed with an arbitrary smooth two-dimensional $T^{1,0}_{\mathrm{II}}\mathfrak{X} \subset \mathbb{C} \otimes T\mathfrak{X}$.

1.6 Q$\mathbb{C}$-Legendrian bundles

Definition 7. A Q$\mathbb{C}$-*Legendrian bundle* $\mathrm{q} : \mathfrak{X} \to Q$ is such smooth mapping from a Q$\mathbb{C}$-contact manifold $\mathfrak{X}$ to a smooth manifold Q that $d\mathrm{q}$ is surjective everywhere and for any $q \in Q$ the "fiber" $\mathfrak{X}_q := \mathrm{q}^{-1}(q)$ is a Legendrian complex curve in $\mathfrak{X}$. An *almost (complex Legendrian) bundle* if the same where $\mathfrak{X}$ is endowed with a smooth $T^{1,0}_{\mathrm{II}}\mathfrak{X}$ of the rank 2 but isn't necessarily Q$\mathbb{C}$-contact.

Remark. Obviously, $\dim Q = \dim_{\mathbb{R}} \mathfrak{X} - 2$. We restricted definition in this way because we defined Q$\mathbb{C}$-contact manifolds only for $m = 1$.

Remark. The Theorem 1 admits some generalization to the Q$\mathbb{C}$-Legendrian case. Although there is no *holomorphic* immersion of Ω_q in full generality, and we can't use the projective complexified *cotangent* bundle for technical reasons, Ω_q has indeed a smooth immersion to $\mathbf{P}(\mathbb{C} \otimes T_q Q)$.

2 Lorentzian manifolds

2.1 The bundle of skies

This subsection sets out the facts known about Lorentzian manifolds (or space-times) along the lines of [2].

Definition 8. A *Lorentzian space-time* is a pseudo-Riemann four-dimensional manifold X with the metric g of the signature $(+---)$ and the *time orientation* at each point $x \in X$ (i.e. one of the two connected components of the cone $\{\, v \in T_x X \mid g(v) > 0 \,\}$ is chosen as "chronological future"), in a continuous fashion.

Unlike [2], X is not assumed to be C^∞ smooth. Our standard smoothness conditions will be C^2 for X proper and also, the metric tensor g must be C^1 and the Levi-Civita connection on TX uniquely integrable.

Definition 9. The sky $\mathfrak{S}_x$ is the base of the cone $\mathcal{C}_x^+$ in T_xX—itself the boundary of the "chronological future" cone—and points on $\mathfrak{S}_x$ will be denoted $\mathrm{P}v$, where $v \in \mathcal{C}_x^+ \setminus \{0\}$. The disjoint union of all skies (of all points of X) forms a smooth locally trivial bundle over X, denoted by $\mathfrak{S}X$, with the projection map $\mathrm{x} : \mathfrak{S}X \to X$.

Definition 10. The *geodesic flow* FX is a distribution of 1-subspaces in the tangent bundle $T(\mathfrak{S}X)$ of the total space of the bundle $\mathfrak{S}X$, defined by the equations $dx \parallel v$ (the differential of x is collinear to v), $\nabla v = 0$ (v is constant under the Levi-Civita connection), where $v \in \mathcal{C}_x^+ \setminus \{0\}$ is a vector representing given point of the sky.

Remark. Integrating the flow FX gives the "light" foliation of the total space $\mathfrak{S}X$. Its leafs are *light lines*, or null geodesic curves on X lifted to $\mathfrak{S}X$ naturally, by the mapping P of tangent vectors to respective skies. The paper[2] also defined notation for the property of elements of $\mathfrak{S}X$ to lie on the same light line, but we don't need this notation here.

From [6], [7], and [4] we know that $\mathfrak{N}$, that is defined as the quotient space of $\mathfrak{S}X$ by the light foliation possesses a natural contact structure (when $\mathfrak{N}$ is smooth). But one doesn't need $\mathfrak{N}$ at all to define the respective distribution of tangent hyperplanes.

Definition 11. The form $\vartheta := (v.dx)$, where $v \in \mathcal{C}^+_{\mathrm{x}(w)} \setminus \{0\}$ is a null vector representing a point $w \in \mathfrak{S}X$ and "." denotes the Lorentz scalar product, defines an 1-form on the total space $\mathfrak{S}X$.

Remark. For each $x \in X$ there is an oriented real line bundle $\mathcal{L}_x^{\mathbb{R}}$ which the values of ϑ on $\mathfrak{S}_x$ pertain to;[4] namely, $\mathcal{L}_x^{\mathbb{R}} \subset \mathcal{O}_{\mathfrak{S}_x}(1,1)$. In fact, these bundles are well-defined *globally* over $\mathfrak{S}X$; refer to [2] for details of the construction.

Remark. The 1-form ϑ is smooth and never vanishes. But ϑ nullifies all the FX (because every Lorentzian null direction is orthogonal to itself) and all vertical (going along the fibers) directions in $\mathfrak{S}X$ because $d\mathrm{x} = 0$. In other words, $T(\mathfrak{S}X)$ has a continuous distribution $H\mathfrak{S}X := \ker \vartheta$ of homogeneous co-oriented hyperplanes, a distribution containing $T\mathfrak{S}_x$ for all $x \in X$.

[4] Here is not important *which* bundle namely it is. One can also use a tautological contact form similar to the form from the subsection 1.5. Alternatively, one may choose for each point $w \in \mathfrak{S}X$ its representative $v \in \mathcal{C}^+_{\mathrm{x}(w)} \setminus \{0\}$ in a smooth fashion, which leads to a nominally real-valued form, but meaningful only up to multiplication by a positive function.

Remark. For a C^2-smooth[5] $\mathfrak{S}X$ for any vertical vector field V on $\mathfrak{S}X$ (that is, a section of $\ker d\mathrm{x} \subset T\mathfrak{S}X$) and any light vector field L (that is, a section of $FX \hookrightarrow T\mathfrak{S}X$), their Lie bracket satisfies $\vartheta[V, L] = 0$.

2.2 Extensions for light lines and $\mathfrak{S}X$

Consider some light line ℓ in $\mathfrak{S}X$. Each its point specifies a plane tangent to the respective sky in $T\mathfrak{S}X$ and, after quotient by the light foliation, in $T_\ell\mathfrak{N}$ ($\mathfrak{N}$ is not necessarily a manifold, but has tangent spaces defined *locally*). All skies are Legendrian, hence all these homogeneous planes actually lie in $T\mathrm{c}_\ell\,\mathfrak{N}$,[6] giving an immersion—and in sane cases also an embedding—of the light line ℓ into $\widetilde{\mathrm{Gr}}_2(T\mathrm{c})$, assuming four-dimensional orientation of the space-time[7] at ℓ. Each sky has a complex structure, hence we can lift it against the complex-forgetful map to $\mathbf{P}(\mathbb{C} \otimes T\mathrm{c}_\ell\,\mathfrak{N}) \setminus \mathbf{P}\ T\mathrm{c}_\ell\,\mathfrak{N}$, denoting the image with $\hat{\ell}$.[8]

In some cases—such as when the space-time X is real analytic, but not necessarily because $\hat{\ell}$ is not necessarily differentiable with respect to the smooth structure of X—it's possible to find a *holomorphic* curve $\hat{U} \supset \hat{\ell}$ in $\mathbf{P}(\mathbb{C} \otimes T\mathrm{c}_\ell\,\mathfrak{N})$.

Definition 12. If, for some light line ℓ, a holomorphic curve $\hat{U} \subset \mathbf{P}(\mathbb{C} \otimes T\mathrm{c}_\ell\,\mathfrak{N})$ contains the lifting $\hat{\ell}$ of the light line constructed above, and $\hat{\ell}$ is closed in $\hat{U}$, then $\hat{U}$ is a [non-degenerate] *lateral extension* for ℓ.

Remark. Points of $\hat{\ell}$ are projectivisations of $T^{1,0}$ vectors tangent to skies on ℓ. Substituting the external derivative of a contact form in $\mathfrak{N}$, restricted to $T\mathrm{c}_\ell\,\mathfrak{N}$, for ω from the subsection 1.3 gives a complex contact bundle over the Grassmanian; we also can check that $\omega(\mathbf{z}, d\mathbf{z}) = 0$ for such $\mathbf{z} \in \mathbb{C} \otimes T\mathrm{c}_\ell\,\mathfrak{N}$ that originate from $T^{1,0}$ spaces of skies along ℓ. Hence $\hat{U}$ is a holomorphic *Legendrian* curve by its property to contain $\hat{\ell}$.

[5] Our standard smoothness condition can guarantee only C^1 for $\mathfrak{S}X$. With C^2 an even stronger relation holds: $[H\mathfrak{S}X, FX] \subset H\mathfrak{S}X$.

[6] The contact subspace $T\mathrm{c}_\ell\,\mathfrak{N} \subset T_\ell\mathfrak{N}$ is the pushforward of $H_\ell\,\mathfrak{S}X$ by the projection from the bundle of skies onto $\mathfrak{N}$.

[7] Or orientation of the space alone – both things are equivalent in this time-oriented context.

[8] Henceforth we shall write "$\hat{\ell} \subset \mathbf{P}(\mathbb{C} \otimes T\mathrm{c}_\ell\,\mathfrak{N})$" and so on by some abuse of notation, although it must be understood that only *immersion* is certain; there is no warranty against self-intersections or multiple covering.

Example. Trivially, for a conformally flat (such as Minkowski space) case—or, more generally, where ℓ has no shear—the respective two-dimensional complex structure[9] forms a lateral extension.

Definition 13. The *tautological bundle* $E\hat{U}$ for a lateral extension $\hat{U}$ is a holomorphic line bundle restricted from the tautological line bundle of $\mathbf{P}(\mathbb{C}\otimes Tc_\ell\,\mathfrak{N})$ to $\hat{U}$.

By $\mathrm{Re} : \mathbb{C}\otimes Tc_\ell\,\mathfrak{N} \to Tc_\ell\,\mathfrak{N}$, the vector-space version of the [projective] complex-forgetful plane map r, the fibers of $E\hat{U}$ [10] are canonically isomorphic to homogeneous real 2-subspaces of $Tc_\ell\,\mathfrak{N}$.

Remark. Over the real part $\hat{\ell}$ of $\hat{U}$ all fibers of this bundle are complex ($T^{1,0}$) tangent lines to skies.

$E\hat{U}$ as a whole is naturally mapped to $Tc_\ell\,\mathfrak{N}$.

We can also admit "degenerate" lateral extensions—those where $\hat{U}$ is not even immersed to $\mathbb{C}\otimes Tc_\ell\,\mathfrak{N}$, but only holomorphically mapped—but better to pass straightly to extension of a whole bundle of skies.

Definition 14. A *massless extension* of the bundle of skies $\mathfrak{S}X$ is a smooth[11] manifold $\mathcal{S}$, $\dim_{\mathbb{R}}\mathcal{S} = 7$, endowed with a smooth co-oriented distribution $H\mathcal{S}$ of tangent hyperplanes, such CR structure $T^{1,0}_{\mathrm{I}}\mathcal{S} \subset \mathbb{C}\otimes H\mathcal{S}$ having one complex dimension that $T^c\mathcal{S} := \mathrm{Re}\,T^{1,0}_{\mathrm{I}}\mathcal{S}$ is integrable[12] and $[H\mathcal{S}, T^c\mathcal{S}] \subset H\mathcal{S}$, a smooth complex line bundle $E\mathcal{S}$ embedded into $\mathbb{C}\otimes(H\mathcal{S}/T^c\mathcal{S})$ holomorphically along the leafs of $T^c\mathcal{S}$,[13] and a continuous mapping $\iota : \mathfrak{S}X \hookrightarrow \mathcal{S}$ satisfying the following conditions:

- the image of ι is closed in $\mathcal{S}$;
- ι is injective;

[9] See [5] subsection 2.1 or [2] Minkowski space for details of the complex embedding for $\mathfrak{N}$—the affine part of a projective manifold $\mathbf{PN}$—in the Minkowski case.

[10] More correctly, the isomorphism takes place only restricted to $\hat{U}\setminus\mathbf{P}(Tc_\ell\,\mathfrak{N})$. All points of $\mathbf{P}(Tc_\ell\,\mathfrak{N})$ can be removed from $\hat{U}$, preserving its property to be a lateral extension.

[11] Can have a C^∞ structure but, in practice, no derivative will be used above the order 2 which is needed to define the Lie bracket of vector fields.

[12] That is, the CR structure results in foliation by complex curves. Not *every* CR structure, of course, admits complex curves.

[13] $T\mathcal{S}/T^c\mathcal{S}$ is trivial along the complex leafs due to integrability of $T^c\mathcal{S}$, and each of these leafs has constant $H\mathcal{S}/T^c\mathcal{S} \subset T\mathcal{S}/T^c\mathcal{S}$ because of $[H\mathcal{S}, T^c\mathcal{S}] \subset H\mathcal{S}$.

- each light line in $\mathfrak{S}X$ maps by ι to *one* complex curve;[14]
- each point of $\mathcal{S}$ can be connected with the image of ι by a complex path;
- for any $w \in \mathfrak{S}X$ there is such neighborhood U_w of its image $\iota(w) \subset U_w \subset \mathcal{S}$ that ι descends to a C^1 diffeomorphism from the quotient of $\iota^{-1}(U_w)$ by the light foliation onto the quotient of U_w by the foliation of complex curves;
- the partial derivative of ι along skies $\mathfrak{S}_x$ for any $x \in X$ exists, is continuous on $\mathfrak{S}X$, and the equivalence class of the complex partial derivative of ι along skies under quotient by $\mathbb{C} \otimes T^c\mathcal{S}$ belongs to $E\,\mathcal{S}$ everywhere.

Definition 15. For a massless extension $\mathcal{S}$ of $\mathfrak{S}X$, denote by $\mathfrak{N}_\mathcal{S}$ the quotient space of $\mathcal{S}$ by its complex leafs.

Remark. In "civilized" cases $\mathfrak{N}_\mathcal{S}$ and $\mathfrak{N}$ coincide. Generally $\mathfrak{N}_\mathcal{S}$, although has a local smooth structure, *may be* non-Hausdorff as well as $\mathfrak{N}$, but in any case a continuous mapping from $\mathfrak{N}$ onto $\mathfrak{N}_\mathcal{S}$ is defined according to their respective definitions and the light-lines mapping property (Definition 14). If $\mathfrak{N}_\mathcal{S}$ happened to be a five-dimensional C^1 manifold[15] globally, then it possesses a co-oriented contact structure descended from $H\mathcal{S}$. In this case—which will be henceforth the main interesting for us—we can do differential geometry relevant to the contact structure on $\mathfrak{N}_\mathcal{S}$ and abandon $\mathfrak{N}$ altogether.

Remark. A massless extension could be thought of as of a fiber bundle over $\mathfrak{N}_\mathcal{S}$ having one-dimensional complex fibers—*complex light lines*—but we don't require different "fibers" to be homeomorphic (let alone biholomorphic), hence better to reuse the concept of a Cauchy–Riemann structure.

Remark. If $\mathfrak{N}_\mathcal{S}$ is a manifold, then $T^c\mathcal{S} = \ker d\mathrm{q}$ where q denotes the quotient mapping to $\mathfrak{N}_\mathcal{S}$. Hence $H\mathcal{S}/T^c\mathcal{S}$ is the same as $\mathrm{q}^*\ Tc\,\mathfrak{N}_\mathcal{S}$, and $E\,\mathcal{S}$ specifies a mapping $\mathrm{P}E : \mathcal{S} \to \mathbf{P}(\mathbb{C} \otimes Tc\,\mathfrak{N}_\mathcal{S})$. On the other hand, $E\,\mathcal{S}$ is a weaker structure than Q$\mathbb{C}$-contact could be. Indeed, were certain $T^{1,0}_{\mathrm{II}}\mathcal{S} \supset T^{1,0}_{\mathrm{I}}\mathcal{S}$, then we could define $E\,\mathcal{S} := T^{1,0}_{\mathrm{II}}\mathcal{S} \mod \mathbb{C} \otimes T^c\mathcal{S}$ that is $E\,\mathcal{S} + \mathbb{C} \otimes T^c\mathcal{S} = T^{1,0}_{\mathrm{II}}\mathcal{S} \oplus T^{0,1}_{\mathrm{I}}\mathcal{S}$ (remind that $\mathbb{C} \otimes T^c = T^{1,0} \oplus T^{0,1}$). But, having only $E + \mathbb{C} \otimes$

[14] That is, an integral submanifold of $T^c\mathcal{S}$. For differentiable case it means $FX \subset \iota^*T^c\mathcal{S}$.

[15] We assume such C^1 structure that the projection from $\mathfrak{S}X$ to $\mathfrak{N}_\mathcal{S}$ is C^1, not just an *arbitrary* C^1 smoothness structure.

T^c as a three-dimensional complex subspace in $\mathbb{C} \otimes H$, there is no unique subtraction operation to take $T_{\mathrm{I}}^{0,1}$ away and obtain a *two*-dimensional $T_{\mathrm{II}}^{1,0}$.

Remark. A massless extension obviously exists for a real analytic X (use a tube-like thing resulting from the union of all complexified null geodesics to construct), producing analytic $\mathcal{S}$ and ι in their turn. The conditions on a massless extension in general can be thought of as a "weak analyticity" condition on X.

Remark. Note that—although *each* sky must be embedded to $\mathcal{S}$ smoothly (C^1)—the "embedding" $\iota : \mathfrak{S}X \hookrightarrow \mathcal{S}$ is only for **topological** manifolds and isn't necessarily a C^1 mapping! In some points of the bundle of skies it may be not differentiable along the geodesic flow (light lines).

2.3 The main theorem

Now we are prepared to describe the structure of $\mathfrak{N}_{\mathcal{S}}$ with (quasi) complex Legendrian bundles. $\mathfrak{N}_{\mathcal{S}}$ (like $\mathfrak{N}$) is *five*-dimensional, hence a Q$\mathbb{C}$-Legendrian bundle for $\dim \mathfrak{X} = 7$ can fit into this base.

Since our aim is to obtain smooth structures continued to the complexified light dimension, we must restrain (with some analyticity conditions) the original data coming from $\mathfrak{S}X$. The subbundle

$$E_{\mathrm{I}} := \bigsqcup_{x \in X} \iota_* \left(T^{1,0}\mathfrak{S}_x\right) \quad \subset \quad \mathbb{C} \otimes H\mathcal{S}|_{\iota(\mathfrak{S}X)}$$

is canonically isomorphic to $E\,\mathcal{S}|_{\iota(\mathfrak{S}X)}$ because fibers of $E\,\mathcal{S}$ coincide with tangent planes to skies on the real bundle of skies. We should impose conditions on dependence of E_{I}—named the *connexion of skies*—on points of $\iota(\mathfrak{S}X)$.

Theorem 2. *Let X be an oriented space-time and $\mathcal{S}$ a massless extension of its bundle of skies $\mathfrak{S}X$. Also suppose that the quotient $\mathfrak{N}_{\mathcal{S}}$ of $\mathcal{S}$ by its complex leafs foliation is a smooth manifold and has such real analytic structure that the connexion of skies E_{I} (a complex line subbundle in $\mathbb{C} \otimes H\mathcal{S}|_{\iota(\mathfrak{S}X)}$) is holomorphic along the complex leafs of $\mathcal{S}$.*[16] *Then there exists such $\mathfrak{X}$, $\iota : \mathfrak{S}X \hookrightarrow \mathfrak{X} \subset \mathcal{S}$—a smaller*[17] *continuation of the bundle of skies—endowed*

[16] $\mathcal{S}$ possesses a natural mapping $\mathrm{P}E$ to $\mathbf{P}(\mathbb{C} \otimes T\mathrm{c}\,\mathfrak{N}_{\mathcal{S}})$. Although we don't require it to be analytic—that is, analyticity for the massless extension—the total space $\mathbf{P}(\mathbb{C} \otimes T\mathrm{c}\,\mathfrak{N}_{\mathcal{S}})$ *is* analytic, and because of embedding of E_{I} into the tangent bundle of the latter manifold, we can judge about partial analyticity of E_{I}.

[17] Possibly $\mathfrak{X} = \mathcal{S}$ as smooth manifolds.

with such smooth $T^{1,0}_{\mathrm{II}}\mathfrak{X} \subset \mathbb{C} \otimes H\mathfrak{X}$ *of the rank 2 (where* $H\mathfrak{X} := H\mathcal{S}|_{\mathfrak{X}}$*) that* $T^{1,0}_{\mathrm{II}}\mathfrak{X}$ *subsumes the CR structure restricted from* $\mathcal{S}$*,*[18] *and embeddings* $\mathfrak{S}_x \hookrightarrow \mathfrak{X}$*,* $x \in X$ *produce almost (complex Legendrian) curves, preserving the complex structure on skies. Moreover, the quotient map* $\mathrm{q} : \mathfrak{X} \to \mathfrak{N}_{\mathcal{S}}$ *(restricted from* $\mathcal{S}$ *to* $\mathfrak{X}$*) forms an almost (complex Legendrian) bundle which is* $\mathrm{Q}\mathbb{C}$*-Legendrian outside* $\mathcal{S}_{\mathrm{deg}}$*, where* $\mathcal{S}_{\mathrm{deg}}$ *is the closed set where the derivative of* $E\,\mathcal{S}$ *along the complex leafs is zero,*[19] *and* $\operatorname{Re} T^{1,0}_{\mathrm{III}}(\mathfrak{X} \setminus \mathcal{S}_{\mathrm{deg}})$*—see subsection 1.5—is the same tangent subbundle as* $H\mathcal{S}|_{\mathfrak{X}\setminus\mathcal{S}_{\mathrm{deg}}}$*.*

Remark. Analiticity for $\mathfrak{N}_{\mathcal{S}}$ is, in fact, "spatial-only analiticity", not an unreasonable thing for "real-world physics" because the standard construction for the space of null geodesics is $\mathbf{S}T'M$ where M is either a Cauchy surface (see e.g. [4]) or the cosmological singularity (see e.g. [2]). Conditions in terms of X are yet short of real analiticity and admit cases of *non-smooth* embedding of it into a complex space-time.

Proof. We have to "upgrade" one-complex-dimensional CR structure of the massless extension to a two-complex-dimensional tangent subbundle structure suitable for a $\mathrm{Q}\mathbb{C}$-complex manifold. For it, we make continuation of the connexion of skies which came from $\mathfrak{S}X$ (and is defined over its image), with its derivatives, to some $\mathfrak{X} \subset \mathcal{S}$, a neighborhood of the image of $\mathfrak{S}X$. The continued connexion of skies—that is, a line subbundle in $\mathbb{C} \otimes H\mathcal{S}$—gives us the second complex dimension. $T^{1,0}_{\mathrm{II}}\mathfrak{X}$ must be the direct sum of $T^{1,0}_{\mathrm{I}}\mathfrak{X} = T^{1,0}_{\mathrm{I}}\mathcal{S}|_{\mathfrak{X}}$ (the complex light direction) with the connexion of skies.

The fact that the Lie bracket, in fact, *adds* one dimension to $T^{1,0}_{\mathrm{II}}$ everywhere on $\mathfrak{X}$ except $\mathfrak{X} \cap \mathcal{S}_{\mathrm{deg}}$—and that $\operatorname{Re} T^{1,0}_{\mathrm{III}}(\mathfrak{X} \setminus \mathcal{S}_{\mathrm{deg}}) = H\mathcal{S}|_{\mathfrak{X}\setminus\mathcal{S}_{\mathrm{deg}}}$ namely—follows from computation of the Lie bracket between sections of E_{I} and $T^{1,0}_{\mathrm{I}}\mathfrak{X}$.

The complex structure of skies is preserved by the construction of $T^{1,0}_{\mathrm{II}}\mathfrak{X}$ from E_{I}. Also by construction, E_{I} and $T^{1,0}_{\mathrm{I}}$ represent in each point of $\iota(\mathfrak{S}X)$ two *real* tangent planes intersecting by $\{0\}$, hence $T^{1,0}_{\mathrm{II}}\mathfrak{X} \cap \mathbb{C} \otimes T(\iota(\mathfrak{S}_x)) = E_{\mathrm{I}}|_{\iota(\mathfrak{S}_x)}$ – that is, $\iota(\mathfrak{S}_x)$ is Legendrian for any $x \in X$.

[18] $T^{1,0}_{\mathrm{II}}\mathfrak{X} \supset T^{1,0}_{\mathrm{I}}\mathfrak{X} := T^{1,0}_{\mathrm{I}}\mathcal{S}|_{\mathfrak{X}}$ or, in integral terms, complex leafs of $\mathcal{S}$ restrict to Legendrian complex paths in $\mathfrak{X}$.

[19] In the important case when the metric tensor g is C^2-smooth everywhere and each leaf of $\mathcal{S}$ is a non-degenerate extension—a curve immersed in $\mathbf{P}(\mathbb{C} \otimes T\mathrm{c}_\ell\,\mathfrak{N}_{\mathcal{S}})$—there is $\mathcal{S}_{\mathrm{deg}} \cap \iota(\mathfrak{S}X) = \emptyset$, hence, in this case, if such massless extension exists at all, then it is always possible without these "degenerate" points.

At the end, fibers $\mathfrak{X}_w := \mathrm{q}^{-1}(w)$ are Legendrian in $\mathfrak{X}$ due to the same $\{0\}$-intersection observation. $\square$

2.4 Concluding remarks

Beyond the first Lie bracket. We arguably may use the (almost) complex geometric metaphor for dimensions from $\operatorname{Re} T^{1,0}_{\mathrm{II}}\mathfrak{X}$ in the context of Theorem 2. As for (generally) three-dimensional $T^{1,0}_{\mathrm{III}}\mathfrak{X} = [T^{1,0}_{\mathrm{II}}\mathfrak{X}, T^{1,0}_{\mathrm{II}}\mathfrak{X}]$, Lie brackets of its sections with complex light vector fields (sections of $T^{1,0}_{\mathrm{I}}$) result in vector fields outside $T^{1,0}_{\mathrm{III}}\mathfrak{X}$ wherever X is not conformally flat. This effect is due to *shear*[20] which—for every $x \in X$ where the Weil curvature is non-zero—is also non-zero on all $\mathfrak{S}_x$ except for four points at most. Hence, no three-complex-dimensional CR structure is possible for extensions of the bundle of skies.

Extra complex dimensions. But we also see an immersion $\mathrm{P}E : \mathfrak{X} \to \mathbf{P}(\mathbb{C} \otimes Tc\,\mathfrak{N}_{\mathcal{S}})$, which should be embedding in reasonable cases. It can be used in the context of gravitation (not only *conformally invariant* Weil curvature, but things dependent of the metric tensor g inside a conformal class as well) and these findings will be published separately.

* * *

This version of the paper is likely to be updated once more in early 2019.[21] Some proofs need improvement and some examples are omitted.

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[20] See [5] section 3 for discussion on geometric effects of shear with respect to the bundle of skies. Note that the authors avoid the word "sky".

[21] Look for real-time updates at `http://course.irccity.ru/celestial/HSoNG.pdf`

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